# (Geometric) representation theory: an introduction

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March 3, 2024

Geometric representation theory is about *using geometry to construct and study representations*. (Of course, representation theory is useful in geometry too! But the term does not usually refer to such applications.) Before we dive into a few of the gems of this theory, we recall some of the basics of representation theory itself.

## 1 Representation theory of Lie groups and Lie algebras

This section is a crash course on Lie groups and algebras and their representation theory. The main idea is to understand *continuous symmetries*, that is, geometric spaces that admit symmetries by groups with a manifold or variety structure. These symmetries are ubiquitous in geometry, algebra, physics, number theory, and many other subjects. (In number theory, these groups are also often replaced by locally compact groups which are totally disconnected, such as $p$-adic and profinite groups, which have many similarities.) I won't be able to discuss all of this in the actual lecture.

### 1.1 Group representations

*Representation theory* is the study of linear symmetry, in its many incarnations. The main notion is:

**Definition 1.1.** A representation of a group $G$ is a pair $(V, \rho_V)$ of a vector space $V$ and a homomorphism $\rho_V : G \to \mathsf{GL}(V)$. It is called *faithful* if $\rho_V$ is injective. It is called *trivial* if $\rho_V(g)$ is the identity for all $g$.

**Definition 1.2.** A map $T : V \to W$ is called a homomorphism of representations, or a $G$-linear map, if $T \circ \rho_V(g) = \rho_W(g) \circ T$ for all $g \in G$.

One can also replace $\mathsf{GL}(V)$ by alternatives, such as the orthogonal or symplectic group (arising in Riemannian and symplectic geometry, respectively):

**Example 1.3.** Suppose that $G$ acts by symmetries on manifold or variety $X$, for example, $X$ is a homogeneous space $G/H$ for $H \leq G$. At every fixed point $x \in X$, $G$ induces a representation $G \to \mathsf{GL}(T_x X)$, by taking the first derivative. If $G$ acts by Riemannian isometry, then the image lies in the orthogonal group $\mathsf{O}(T_x X)$, and if it acts by symplectic isometry, the image lies in the symplectic group $\mathsf{Sp}(T_x X)$.

**Example 1.4.** Continuing with $G$ acting by symmetries on a space $X$, there are other representations: for example, $G$ acts on the cohomology $H^*(X)$. For any natural linearisation (approximation of $X$ by a vector space), $G$ acts on it.

**Example 1.5.** If $G$ is a symmetry group of a system of differential equations, then $G$ also acts on the vector space of solutions. So for example, the group of 3D rotations acts on solutions of the Schrödinger equation for the hydrogen atom (which has spherical symmetry), decomposing them into the various spherical harmonics (orbitals, energy levels, etc.).

## 1.2 Lie groups

**Definition 1.6.** A *Lie group* is a group which is also a manifold, such that the multiplication and inversion are smooth operations. A representation is a map $\rho_V : G \to \mathsf{GL}(V)$ which is smooth (with $V$ a real or complex vector space). A *complex Lie group* is a group which is a complex manifold, and a representation is a holomorphic map $\rho_V : G \to \mathsf{GL}(V)$ with $V$ a complex vector space. A representation is called *orthogonal* (in the real case) or *unitary* (in the complex case) if it preserves an inner product (so, has target $O(V)$ or $U(V)$, respectively).

For Lie groups, a representation $\rho_V : G \to \mathsf{GL}(V)$ is assumed to be a Lie group homomorphism, i.e., the homomorphism should be a smooth map. Lie groups can be real or complex, in the latter case we assume that they are complex manifolds with holomorphic multiplication and inversion. One can alternatively use *algebraic groups*, where $G$ is an algebraic variety instead of a manifold. The advantage of this is that one can work over any field, not just $\mathbb{R}$ or $\mathbb{C}$. Indeed, algebraic groups over $\mathbb{R}$ or $\mathbb{C}$ are in particular Lie groups. For sake of accessibility, we will stick to the setting of Lie groups in these notes, but many assertions have analogues for the algebraic case.

To make proper sense of the smoothness, we will mostly restrict to the case that $V$ is finite-dimensional. More generally, we can let $V$ be a Hilbert space (or Banach, or complete locally convex topological vector space, if we don't require unitarity).

**Example 1.7.** Lie groups include, of course, $\mathsf{GL}(V)$ ("general linear group") for any finite-dimensional vector space $V$ over $\mathbb{R}$ or $\mathbb{C}$. They also include $\mathsf{SL}(V)$ ("special linear group"), the subgroup of determinant one transformations.

**Example 1.8.** If $V$ is equipped with a symmetric bilinear form then the group preserving it is the orthogonal group $\mathsf{O}(V)$. In the case of $V = k^n$ with $k$ a field and the standard dot product, we get the group $\mathsf{O}_n(k)$ (e.g., $k = \mathbb{R}, \mathbb{C}, \ldots$). For $k = \mathbb{R}$ we have also the famous groups $\mathsf{O}_{p,q}(\mathbb{R})$ preserving the form $(a_1, \ldots, a_{p+q}, b_1, \ldots, b_{p+q}) = a_1 b_1 + \cdots + a_p b_p - a_{p+1} b_{p+1} - \cdots - a_{p+q} b_{p+q}$. These groups are all Lie for $k = \mathbb{R}, \mathbb{C}$.

**Example 1.9.** Similarly, if $V$ is equipped with a symplectic form the group preserving it is $\mathsf{Sp}(V) \leq \mathsf{GL}(V)$. For the standard symplectic form on $k^n$ we get $\mathsf{Sp}_n(k)$. Regardless of $k$

there is only one of these up to equivalence for each finite dimension of $V$. For $k = \mathbb{R}, \mathbb{C}$ this is a Lie group.

**Example 1.10.** Any abstract (not topological/geometric) group can be given a Lie group structure by considering it to be a discrete topological space (hence a zero-dimensional manifold, possibly with infinitely many connected components).

## 1.3 The four main building blocks of connected Lie groups

In this section we consider representations of the four main building blocks of connected Lie groups: the circle $S^1$, the line $\mathbb{R}$, the three-sphere $S^3 \cong \mathsf{SU}_2$, and the matrix group $\mathsf{SL}_2(\mathbb{R})$. Along the way we also discuss $\mathsf{SO}_3(\mathbb{R})$. A useful concept is that of an *irreducible* representation:

**Definition 1.11.** A *subrepresentation* of $(V, \rho_V)$ is a subspace $W \subseteq V$ invariant under $G$. The representation $(V, \rho_V)$ is *irreducible* if the only (closed) subrepresentations are 0 and $V$. A finite-dimensional representation is *semisimple* if it is a direct sum of irreducible representations.

**Remark 1.12.** (Technical:) The word "closed" is redundant in the finite-dimensional case, but very important in the infinite-dimensional case, where the representations are usually spaces of functions $V$, and one can construct many different types of functions (analytic, smooth, distributions, etc.), each of which are dense subrepresentations, so we don't want this to count against irreducibility.

**Example 1.13.** The circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, viewed as the unit circle in the complex plane under multiplication. This acts on $\mathbb{C}$ by multiplication, giving a one-dimensional representation $S^1 \to \mathsf{GL}(\mathbb{C}) = \mathbb{C}^\times$. If we view $\mathbb{C}$ alternatively as $\mathbb{R}^2$, we get the group of rotations fixing the origin, $\mathsf{SO}(2) := \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} |a^2 + b^2 = 1\}$.

There are lots of one-dimensional complex representations of $S^1$, given by $\chi_n : z \mapsto z^n \in \mathbb{C}^\times = \mathsf{GL}_1(\mathbb{C})$ for $n \in \mathbb{Z}$ (these can also be viewed as two-dimensional real representations). It turns out that these are all the irreducible representations. Moreover, every finite-dimensional representation $(V, \rho_V)$ is isomorphic to $\chi_{n_1} \oplus \cdots \oplus \chi_{n_m}$ for some $n_i \in \mathbb{Z}$, which means that in some basis, $\rho_V(z)$ becomes a diagonal matrix with entries $z^{n_1}, \ldots, z^{n_m}$. In fact the same is true for infinite-dimensional representations if we take a completion of the direct sum.

This is closely related to the Fourier transform: let $V = \mathrm{Fun}(S^1, \mathbb{C})$ be a vector space of functions $S^1 \to \mathbb{C}$ (e.g., smooth, continuous, or square-integrable). We have an action on $V$ of $S^1$ by $(\lambda \cdot f)(z) = f(\lambda z)$. We then have a subrepresentation $\mathbb{C} \cdot e^{2\pi i m z}$, on which the action is $\rho_V(\lambda)(v) = \lambda^m v$. So the decomposition of $V$ into its one-dimensional (irreducible) representations is the Fourier transform. Indeed, a function $f$ can be represented as $f = \sum_{m \in \mathbb{Z}} \hat{f}_m e^{2\pi i m z}$ for $\hat{f}_m$ the Fourier coefficient $\hat{f}_m = \frac{1}{2pi} \int_{S^1} e^{-2\pi i m x} f(x) dx$.

**Example 1.14.** As a stark contrast, we can replace $S^1$ by its universal cover, $\mathbb{R}$. We have the corresponding one-dimensional representations as before, $x \mapsto e^{2\pi imx} \in \mathbb{C}^\times$ for $m \in \mathbb{Z}$. But now we have lots more representations: we can replace $m$ with any complex number. Moreover, we can replace it by an $n \times n$ matrix $A$: $x \mapsto e^{2\pi iAx}$. It turns out these are all of the finite-dimensional representations. By linear algebra we can conjugate $A$ to an upper-triangular matrix, but not necessarily to a diagonal one. Thus, these need not be sums of one-dimensional representations, but they admit filtrations $0 \subsetneq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$ for $n = \dim V$, with one-dimensional quotients; in particular the irreducible finite-dimensional representations are the one-dimensional ones mentioned, $x \mapsto e^{2\pi i\lambda x}$ for $\lambda \in \mathbb{C}$, but general finite-dimensional representations need not be direct sums of these.

Moreover, unlike for $S^1$, we can have irreducible infinite-dimensional representations: in 1976, Per Enflo constructed a bounded operator $T : E \to E$ on an infinite-dimensional complex Banach space $V$ such that 0 and $V$ are the only closed $T$-invariant subspaces. Therefore, $x \mapsto e^{2\pi iTx}$ affords an infinite-dimensional irreducible representation $V$ of $\mathbb{R}$. (The question of whether this exists for a Hilbert space is possibly still open, with a preprint of Enflo appearing last year claiming to resolve this in the negative!)

**Example 1.15.** The simply-connected compact Lie group of smallest dimension is the three-sphere $S^3$, which can be given a group structure in at least three different ways (all producing an equivalent answer): the group of unit quaternions $S^3 \subseteq \mathbb{H}$, the group $\mathsf{SU}_2 := \left\{ \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \,\middle|\, |a|^2 + |b|^2 = 1 \right\}$ of special unitary matrices (matrices of determinant one with $A = (\overline{A}^t)^{-1}$), or alternatively as the spin group $S^3 \cong \mathsf{Spin}_3(\mathbb{R})$, the universal (two-to-one) covering of the group $\mathsf{SO}_3(\mathbb{R})$ of rotations of $\mathbb{R}^3$ fixing the origin. The covering map $S^3 \to \mathsf{SO}_3(\mathbb{R})$ sends the unit quaternion $q \in S^3 \subseteq \mathbb{H}$ to the orthogonal transformation $x \mapsto qxq^{-1}$ of the imaginary quaternions $\mathbb{R}^3 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \subseteq \mathbb{H}$.

It turns out that the irreducible complex representations of $S^3$ are all finite-dimensional, and there is one of every dimension up to isomorphism: the $(m+1)$-dimensional representations $\mathbb{C}[x,y]_m$ of degree $m \geq 0$ polynomials in two variables $x, y$, with action given by the multiplication of $\mathsf{SU}_2$ on $\mathbb{C}^2$, namely, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x,y) = f(ax+cy, bx+dy)$. The representation $\mathbb{C}[x,y]_0 \cong \mathbb{C}$ is trivial. Viewing these as real representations, one can see that for even $m$, they split into two isomorphic real representations of odd dimension $m+1$, whereas for odd $m$, $\mathbb{C}[x,y]_m$ is also irreducible as a real representation of dimension $2(m+1)$ (a multiple of 4). So the dimensions of the real irreducible representations are the odd positive integers and the positive integer multiples of four.

Moreover, just as for $S^1$, all irreducible representations are isomorphic to direct sums of these (or a completion thereof, in the infinite-dimensional case).

**Example 1.16.** Bonus example: as we said, there is a double covering $S^3 = \mathsf{Spin}_3(\mathbb{R}) \to \mathsf{SO}_3(\mathbb{R})$. The kernel is $\{\pm 1\}$. Therefore, the representations of $\mathsf{SO}_3(\mathbb{R})$ are the representations of $S^3$ which are trivial on $-1$. These are precisely the complex representations $\mathbb{C}[x,y]_{2m}$ in even degree above, of odd dimension $2m+1$, so the real representations are also of odd dimension $2m+1$, the real forms of these. Note that for $m = 1$ we just get the standard three-dimensional representation $\mathbb{R}^3$, which is irreducible. For $m = 2$ we get the five-dimensional representation which, together with the trivial representation, gives the decomposition of $\wedge^2\mathbb{R}^3$ into irreducible representations.

**Example 1.17.** Finally consider the representation theory of the noncompact, non-simply connected group $\mathsf{SL}_2(\mathbb{R})$. Similarly to before, the complex irreducible representations are $\mathbb{C}[x, y]_m$ for $m \geq 0$, but now the real irreducible representations are instead $\mathbb{R}[x, y]_m$. It turns out that all finite-dimensional representations are direct sums of these. However, unlike the case of the compact group $\mathsf{SU}_2$, there are infinite-dimensional irreducible representations, even unitary ones. The unitary ones have a nice property called "admissibility": the maximal compact subgroup $\mathsf{SO}_2(\mathbb{R}) \cong S^1$ acts with each irreducible representation appearing with finite multiplicity. (Wolfgang Soergel also constructed weird inadmissible irreducible representations of $\mathsf{SL}_2(\mathbb{R})$, by inducing an infinite-dimensional irreducible representation of $\mathbb{R}$.) Also, unlike $S^3$, $\mathsf{SL}_2(\mathbb{R})$ is not simply-connected; its fundamental group is $\mathbb{Z}$. Already the two-to-one covering (unique up to isomorphism), called the "metaplectic group", carries an important infinite-dimensional irreducible unitary representation which is not a representation of $\mathsf{SL}_2(\mathbb{R})$, called the "Weil representation", on square-integrable functions on $\mathbb{R}$ ($L^2(\mathbb{R})$), in which a lift of the element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts via the Fourier transform.

We summarise the situation as follows:

1. For the compact (but non-simply connected) abelian group $S^1$, there are a $\mathbb{Z}$-worth of irreducible representations, which are all one-dimensional, and every finite-dimensional representation is a direct sum of these.

2. For the noncompact (but simply connected) abelian group $\mathbb{R}$, there are a $\mathbb{C}$ worth of one-dimensional representations and all finite-dimensional representations can be obtained by *nontrivial* extensions of these. There are also crazy (nonunitary) infinite-dimensional irreducible representations.

3. For the compact, simply-connected, nonabelian group $S^3$, all irreducible representations are finite-dimensional, but now they need not be one-dimensional: there is one complex one for each dimension up to isomorphism. As real representations, the odd-dimensional ones split into two isomorphic irreducible real representations, of dimensions $1, 3, 5, 7, \ldots$, whereas the even-dimensional ones remain irreducible, of dimensions $4, 8, 12, \ldots$. All finite-dimensional representations are direct sums of these. We also discussed the case $\mathsf{SO}_3(\mathbb{R}) \cong S^3/\{\pm 1\}$, and for these we just restrict only to the odd-dimensional irreducibles of $S^3$.

4. For the noncompact group $\mathsf{SL}_2(\mathbb{R})$, the finite-dimensional complex representations are the same as for $S^3$ (and the real ones are almost the same, now having irreducibles of every real dimension), but there are infinite-dimensional irreducible unitary representations (and crazy nonunitary ones). Nontrivial covers have the same finite-dimensional representations, but have even more infinite-dimensional representations.

We will explain this behaviour more in the next sections. We will also make some comments to explain in what sense *all* Lie groups are (almost) built out of the above ones (excluding compact complex groups, such as the "abelian varieties" like elliptic curves; all representations of these are trivial, since all global functions on the group are constant).

## 1.4 (Lie) algebra representations

One can also look at linear actions of other objects than groups.

**Definition 1.18.** An associative algebra is a ring which is a vector space together with a linear multiplication (assumed to be associative and unital, but not commutative). Then, a representation of an algebra $A$ is a pair $(V, \rho_V)$ of a vector space $V$ and an algebra homomorphism $A \to \mathsf{End}(V)$.

A representation of an algebra is just a module over the algebra (or ring). Examples include algebras of differential operators acting on spaces of functions, and algebras of matrices acting on vectors. We won't go too much into this.

**Definition 1.19.** A Lie algebra is a vector space $\mathfrak{g}$ together with a skew-symmetric bilinear operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$. A representation of a Lie algebra $\mathfrak{g}$ is a pair $(V, \rho_V)$ of a vector space $V$ and a Lie algebra homomorphism $\mathfrak{g} \to \mathsf{End}(V)$, where $[S,T] := S \circ T - T \circ S$ for $S, T \in \mathsf{End}(V)$.

Lie algebras were discovered by Sophus Lie in the 1870s, and independently by Wilhelm Killing in the 1880s; the name "Lie algebra" was given by Hermann Weyl in the 1930s, before which "infinitesimal group" as used. They arise as structures on the *tangent space* of the identity of a Lie group. An equivalent construction is to take the vector space of all vector fields on the group invariant under left translation by group actions (or alternatively, under right translation), using the famous Lie bracket on vector fields.

**Example 1.20.** The Lie algebra $\mathfrak{g}$ with zero Lie bracket is called an *abelian* Lie algebra. This is clearly a Lie algebra. It is called abelian because, if $G$ is a connected Lie group, then $G$ is abelian if and only if $\mathfrak{g}$ is.

**Example 1.21.** Consider the one-dimensional Lie algebra $\mathfrak{g} \cong \mathbb{R}$, necessarily abelian. Representations $\rho_V : \mathfrak{g} = \mathbb{R} \to \mathsf{End}(V)$ just correspond to a choice of operator $\rho_V(1) \in \mathsf{End}(V)$.

**Example 1.22.** For the general abelian Lie algebra, a representation $\rho_V : \mathfrak{g} \to \mathsf{End}(V)$ corresponds to a choice of commuting operators, the subspace $\rho_V(\mathfrak{g})$. Concretely, if $\mathfrak{g}$ has a basis $x_1, \ldots, x_n$ then we choose commuting operators $T_i = \rho_V(x_i) \in \mathsf{End}(V)$, and conversely this uniquely determines $\rho_V$.

**Example 1.23.** Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, which has a basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, satisfying the relations $[e,f] = h, [h,e] = 2e, [h,f] = -2f$.

Now let $(V, \rho_V)$ be a finite-dimensional representation. An important exercise is to show that, if $v$ is an eigenvector of $\rho_V(h)$ of eigenvalue $\lambda$, then $\rho_V(e)v$ and $\rho_V(f)v$ are also eigenvectors of eigenvalues $\lambda+2$ and $\lambda-2$, respectively. The same is true replacing eigenvectors by generalised eigenvectors ($(\rho_V(h)-\lambda)^N v = 0$), and we conclude that $\rho_V(e)$ and $\rho_V(f)$ are nilpotent, i.e., $\rho_V(e)^N = 0 = \rho_V(f)^N$ for some $N \geq 1$. Amazingly, it is true that $\rho_V(h)$ is always diagonalisable with integer eigenvalues, and its spectrum (=collection of eigenvalues with multiplicity) uniquely determine $V$ up to isomorphism. In fact, $V$ is isomorphic to a direct sum of representations where the spectrum of $\rho_V(h)$ has the form $-m, 2-m, \ldots, m-2, m$ for some nonnegative integer $m$. Such a representation can be constructed by the polynomials $\mathbb{C}[x,y]_{m-1}$ of degree $m-1$ in two variables $x, y$: indeed, $h \cdot x^a y^b = (a-b)x^a y^b$, so the monomials form the eigenvectors of $h$ of the promised eigenvalues.

Observe that the Lie algebra of a group $G$ only depends on the component $G^0 \leq G$ containing the identity, and if $\tilde{G} \to G$ is a homomorphism which is a covering map, then the Lie algebras of $\tilde{G}$ and of $G$ are isomorphic. (In fact, a topological covering map $\tilde{G} \to G$ of a Lie group $G$ automatically endows $\tilde{G}$ with a unique Lie group structure for which the map is a homomorphism.)

The general relationships between Lie algebras and Lie groups are summarised by the following basic theorems (Lie's theorems):

**Theorem 1.24.** *Taking the derivative yields a map $\mathsf{Hom}_{\textit{Lie groups}}(G, H) \to \mathsf{Hom}_{\textit{Lie algebras}}(\mathfrak{g}, \mathfrak{h})$, which is injective if $G$ is connected and bijective if $G$ is simply connected.*

**Theorem 1.25.** *Every Lie algebra is the Lie algebra of a unique simply-connected Lie group up to unique isomorphism.*

Thus, one could think of the theory of finite-dimensional Lie algebras as the theory of *simply connected Lie groups*.

**Example 1.26.** By Theorem 1.24, Example 1.23 yields a classification of finite-dimensional complex representations of $G = \mathsf{SL}_2(\mathbb{C})$, a simply-connected complex Lie group with $\mathfrak{sl}_2(\mathbb{C})$ as its Lie algebra. They are just direct sums of the irreducible representations, $\mathbb{C}[x,y]_m$ (of dimension $m+1$). A similar statement holds for all complex representations (see Corollary 1.35).

**Example 1.27.** We can also deduce the structure of all finite-dimensional representations of $\mathsf{SU}_2$ nd $\mathsf{SL}_2(\mathbb{R})$ from this. Although the Lie algebras $\mathsf{SU}_2$ and of $\mathsf{SL}_2(\mathbb{R})$ are not isomorphic, we can consider their complexifications $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. These turn out both to be isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. So the complex representations of $\mathfrak{su}_2$ and $\mathfrak{sl}_2(\mathbb{R})$ are the same as the complex-linear representations of $\mathfrak{sl}_2(\mathbb{C})$. Thus we deduce their finite-dimensional complex representations from Example 1.23. We can deduce from this without too much work the finite-dimensional real representations. Since $\mathsf{SU}_2$ is simply-connected, we also deduce the structure of its finite-dimensional representations. Since $\mathsf{SL}_2(\mathbb{R})$ is only connected, we only know *a priori* that their representations are subset of those of $\mathfrak{sl}_2(\mathbb{R})$. However, the later are direct sums of $\mathbb{C}[x,y]_m$ (the complex case) and $\mathbb{R}[x,y]_m$ (the real case), and we can define them all for $\mathsf{SL}_2(\mathbb{R})$.

All this is useful because we have seen that Lie algebras are in some ways simpler than Lie groups, and don't see (all of) the topology of the former. Indeed, Lie algebras are just vector spaces. This means that it is much easier to deal with infinite-dimensional algebras and infinite-dimensional representations. Many examples of infinite-dimensional Lie algebras arise naturally in geometry, such as the vector space of all vector fields on a smooth manifold of positive dimension under the Lie bracket. Of course, sometimes making use of the geometry of the group $G$ can be helpful, so this can go both ways.

## 1.5 Fundamental classification results

There are too many fundamental results about groups, (Lie) algebras, and their representations to put here (since this is supposed to be an introduction to *geometric* representation theory), so we just focus on some results which provide useful context for what will follow.

**Theorem 1.28.** *If $G$ is a compact (real) Lie group, $(V, \rho_V)$ a representation, and $W \subseteq V$ a closed subrepresentation, then there is a complementary subrepresentation $U \subseteq V$, with $V = W \oplus U$.*

A simple induction on dimension produces the consequence:

**Corollary 1.29.** *Every finite-dimensional representation of a compact Lie group is semisimple.*

Let's explain the proof of the theorem. First, if $(V, \rho_V)$ is unitary, we can simply let $W := U^\perp$, and the proof is immediate. This includes all finite-dimensional cases by the following lemma:

**Lemma 1.30.** *If $(V, \rho_V)$ is a finite-dimensional representation of a compact Lie group, then there is an inner product on $V$ for which $\rho_V$ is unitary (or orthogonal).*

*Proof.* Let $\langle -, - \rangle$ be an inner product on $V$ (for $V$ real, this is a positive-definite symmetric bilinear form, and for $V$ complex, it is a positive-definite Hermitian form). Then we can average this over $G$, producing a new form $(v, w) := \mathrm{vol}(G)^{-1} \int_G \langle gv, gw \rangle dg$. The result is $G$-invariant, and still an inner product. $\square$

**Remark 1.31.** This statement is far from true for noncompact groups. For example, the natural two-dimensional complex representation of $\mathsf{SL}_2(\mathbb{R}) \subseteq \mathsf{GL}_2(\mathbb{C})$ is not unitary: not only is $\mathsf{SL}_2(\mathbb{R})$ not a subgroup of $\mathsf{SU}_2$, but no conjugate is, i.e., there exists no $\mathsf{SL}_2(\mathbb{R})$-invariant inner product on the representation. More generally, the only unitary finite-dimensional representations of $\mathsf{SL}_2(\mathbb{R})$ are trivial, using a deep fact saying that the image of $\rho_V : G \to \mathsf{GL}(V)$ is closed for $G$ semisimple and $V$ finite-dimensional: I leave it as an exercise to make the deduction from the deep fact.

Now we proceed to the general proof of the theorem:

*Proof of Theorem 1.28.* Given a subrepresentation $W \subseteq V$, take a linear projection $T : V \to V$ with image $W$ (i.e., $T|_W = I_W$). Then the average $T' = \int_G \rho_V(g) \circ T \circ \rho_V(g)^{-1}$ of $T$ over $G$ is a $G$-linear projection $T'$ with image $W$. The kernel $U := \ker(T')$ is a subrepresentation by $G$-linearity, and $V = U \oplus W$ since $T'$ is a linear projection. $\square$

**Remark 1.32.** The same proof is often given in first courses on group representation theory, for the case $G$ is finite, where it is called Maschke's theorem.

**Corollary 1.33.** *If $\mathfrak{g}$ is the Lie algebra of a simply-connected compact Lie group, then every finite-dimensional representation of $\mathfrak{g}$ is semisimple.*

Another amazing fact about compact Lie groups is that *all* of their irreducible representations are finite-dimensional. In fact a much stronger statement is true:

**Theorem 1.34.** *Let $G$ be a compact group and $(V, \rho_V)$ a nonzero representation. Then $V$ has a nonzero finite-dimensional subrepresentation.*

*Sketch of proof.* Let's just discuss the unitary case, with $V$ a Hilbert space. If we had a $G$-linear, compact operator $T : V \to V$, then the spectral theorem would give that there is an eigenvalue with finite-dimensional eigenspace, which is then a finite-dimensional subrepresentation by $G$-linearity. To construct such an operator, start with an arbitrary nonzero compact operator $T : V \to V$, e.g., $T(u) = \langle u, v \rangle v$ for nonzero $v \in V$, then average it over the group to form $T' := \int_G \rho_V(g) \circ T \circ \rho_V(g)^{-1}$. Then $T'$ is a $G$-invariant compact endomorphism as desired. $\square$

Zorn's lemma plus the preceding theorems implies the even stronger one:

**Corollary 1.35.** *Every representation of $G$ is a completion of a direct sum of irreducible finite-dimensional representations. If the representation is unitary, the direct sum can be taken to be an orthogonal one (distinct summands are orthogonal).*

All of the above put together means that, for compact groups, we are left only to understand the finite-dimensional irreducible representations.

Next, to understand a general compact group, we can try to understand the two basic cases of connected compact groups and finite ones. We are interested in the connected case here. This can further be reduced to the case of the simply-connected ones (which are equivalent to their Lie algebra) and to tori ($(S^1)^n$):

**Theorem 1.36.** *Every connected compact Lie group has a finite covering which is the product of a torus with a simply-connected compact Lie group.*

**Corollary 1.37.** *The Lie algebra of a compact Lie group is the direct sum of an abelian Lie algebra with the Lie algebra of a simply-connected compact Lie group.*

So far, we have discussed only real Lie groups. We could talk about compact complex Lie groups, but their representation theory is trivial, since for $G$ complex compact, a holomorphic function $\rho : G \to \mathsf{GL}(V)$ must be constant, so only the trivial representation is possible.

Instead there is an operation of *complexification* $G \to G_{\mathbb{C}}$ of a real Lie group $G$, which is the universal homomorphism which induces the complexification map $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ on Lie algebras. It has the following property:

**Definition 1.38.** A unipotent Lie group is one which is isomorphic to a subgroup of the group of upper-triangular matrices with 1's on the diagonal.

**Theorem 1.39.** *The following are equivalent for a complex Lie group $G$:*

1. *$G$ is the complexification of a compact Lie group;*
2. *$G$ admits a faithful finite-dimensional representation (i.e., with $\rho_V$ injective) which is semisimple;*
3. *$G$ admits a faithful finite-dimensional representation and $G$ contains no non-trivial normal connected unipotent subgroup.*

**Definition 1.40.** Such a complex Lie group is called *reductive*. A real Lie group is called reductive if its complexification is reductive. A reductive group is called *semisimple* if its centre is finite.

**Remark 1.41.** Having a faithful finite-dimensional representation $(V, \rho_V)$ says that $G$ can be viewed as a subgroup of $\mathsf{GL}(V)$, called a linear group. Theorem 1.39 basically implies that the study of linear complex groups reduces to the two basic cases of the reductive complex Lie groups and the unipotent ones. Unipotent complex groups are extensions of copies of the additive group $\mathbb{C} = \mathbb{R}_{\mathbb{C}}$. There is an extensive theory of reductive groups, explaining how to build them all up from complex tori $\mathbb{C}^\times = S^1_{\mathbb{C}}$ and from $\mathsf{SL}_2(\mathbb{C}) = S^3_{\mathbb{C}} = \mathsf{SL}_2(\mathbb{R})_{\mathbb{C}}$. So this is the sense in which all linear complex groups are built out of $\mathbb{C}, \mathbb{C}^\times, \mathsf{SL}_2(\mathbb{C})$, and this is why we viewed the basic real examples as the groups $\mathbb{R}, S^1, S^3, \mathsf{SL}_2(\mathbb{R})$ whose complexifications are these ones.

**Corollary 1.42.** *Every finite-dimensional representation of a reductive Lie group is semisimple.*

**Remark 1.43.** The same does not hold for the Lie algebras of reductive Lie groups, since these include the abelian one $\mathbb{R}$ (Example 1.14)!

Applying Theorem 1.36, we see that every complex semisimple Lie group admits a finite covering which is the complexification of a simply-connected compact Lie group. From this follows however:

**Corollary 1.44.** *Every finite-dimensional representation of the Lie algebra of semisimple Lie group is semisimple.*

Such Lie algebras are called semisimple, in view of the following result:

**Theorem 1.45.** *A finite-dimensional Lie algebra is the Lie algebra of a semisimple Lie group if and only if it is the direct sum of nonabelian simple Lie algebras (simple meaning there is no nonzero ideal).*

**Definition 1.46.** A finite-dimensional Lie algera is called semisimple if it satisfies one of the equivalent conditions of this theorem. It is called reductive if it is a direct sum of a semisimple Lie algebra and an abelian Lie algebra.

**Example 1.47.** Note that a real Lie group can be reductive but not compact. For example, $\mathsf{SL}_2(\mathbb{R}) = \{A \in \mathsf{GL}_2(\mathbb{R}) \mid \det A = 1\}$ has complexification which is $\mathsf{SL}_2(\mathbb{C})$, just like $\mathsf{SU}_2 \cong S^3$, which is compact. But $\mathsf{SL}_2(\mathbb{R})$ itself is not compact. These are however all semisimple. So they and their Lie algebras have all finite-dimensional representations semisimple.

# 2 The Peter–Weyl theorem

Let $G$ be a compact group. Then we can consider the vector space $L^2(G) := \{\phi : G \to \mathbb{C} \mid \int_G |\phi|^2 \mathrm{dg} < \infty\}$ of square-integrable functions. This is a Hilbert space with the inner product $\langle \phi, \psi \rangle = \int_G \phi(g)\overline{\psi(g)}\mathrm{dg}$. The group $G$ acts on $L^2(G)$, the *right regular representation*, by $g \cdot \phi(h) := \phi(hg)$. This is a unitary action: $\langle g \cdot \phi, g \cdot \psi \rangle = \langle \phi, \psi \rangle$. So this is a very natural unitary representation. The Peter–Weyl theorem elucidates its structure, which turns out to encode all the irreducible representations in a natural way.

**Remark 2.1.** Finite groups are special cases of compact groups. If you saw representation theory of them, you would have seen the Peter–Weyl theorem, which says that every irreducible representation occurs in the regular representation, with multiplicity equal to its dimension. (There every function on $G$ is square-integrable, so $L^2(G)$ is the space of al functions on $G$. But the regular representation is often defined dually as $\mathbb{C}[G] := \{\sum_{g \in G} a_g g : a_g \in \mathbb{C}\}$. Such a definition is less convenient in the case of infinite groups $G$.)

Let $(V, \rho_V)$ be a finite-dimensional representation of $G$. For every element $T \in \mathsf{End}(V)$, we can associate to this the function

$$\varphi_{V,T} : G \to \mathbb{C}, \varphi_{V,T}(g) = \mathsf{tr}(T\rho_V(g)),$$

called a *matrix coefficient*, because if we pick a basis for $V$ and let $T$ be an elementary matrix in this basis, $\varphi_{V,T}(g)$ is literally a coefficient of the matrix of $\rho_V(g)$. This induces a map

$$\mathsf{End}(V) \to L^2(G), \tag{1}$$

which is a morphism of representations if we equip $\mathsf{End}(V) \cong V^* \otimes V$ with the natural $G$ action,

$$\rho_{\mathsf{End}(V)}(g)(T) := \rho_V(g) \circ T \circ \rho_V(g)^{-1}.$$

**Theorem 2.2.** *Let $G$ be a compact group. Then the following hold:*

1. *The matrix coefficients $\varphi_{V,T}$ are dense in $L^2(G)$;*

*2. The representation $L^2(G)$ splits as a completed direct sum of matrix coefficients of finite-dimensional representations: taking the sum over all irreducible finite-dimensional representations of $G$ up to isomorphism, we have an injection*

$$\bigoplus_V \mathsf{End}(V) \hookrightarrow L^2(G)$$

*with dense image.*

In fact, this is a morphism of $G \times G$ representations, where we equip $\mathsf{End}(V)$ and $L^2(G)$ with the actions

$$(g,h) \cdot T = \rho_V(g) \circ T \circ \rho_V(h)^{-1}, \quad ((g,h) \cdot \phi)(k) = \phi(h^{-1}kg).$$

In this interpretation, the $\mathsf{End}(V)$ summands are irreducible $G \times G$ representations, and all nonisomorphic.

**Remark 2.3.** The injectivity in the second part of the theorem follows from the "density theorem", a basic result in representation theory of algebras, which states that the image $\rho_V(g) \subseteq \bigoplus_{i=1}^n \mathsf{End}(V_i)$ is a spanning set for every collection of nonisomorphic finite-dimensional irreducible representations $V_1, \ldots, V_n$. (I teach this to third-year students, and can provide the notes if you like).

We don't have much time to discuss the proof of this theorem:

*Ideas of the proof of Theorem 2.2.* Given two nonisomorphic closed irreducible subrepresentations $V, W$ of $L^2(G)$, we claim that $V$ and $W$ are orthogonal. First note that, by the inner product, we have $W \cong W^*$. Then the inner product defines a map $V \to W^* \cong W$, which must be zero, otherwise the image is isomorphic to $V$ (by irreducibility), hence neither zero nor $W$. This proves the claim. The same argument works using the $G \times G$ action, so we get that the different summands $\mathsf{End}(V)$ are all orthogonal, for nonisomorphic $V$. This proves the injectivity in the second part.

It remains to prove the first part, which gives the density in the second part. Let $W \subseteq L^2(G)$ be the span of all the matrix coefficients. If $W$ is not dense, then $W^\perp \subseteq L^2(G)$ is nonempty. By Theorem 1.34, $W^\perp$ has a finite-dimensional subrepresentation, call it $U$. Then $U \subseteq L^2(G)$ is orthogonal to its space of matrix coefficients $\varphi_{U,T}$. This leads to a contradiction. (In more detail, $L^2(G)$ is equipped with a convolution, $\varphi * \psi(h) = \int_G \varphi(hg^{-1})\psi(g)\mathrm{d}g$, satisfying $\langle u * v, w \rangle = \langle u, v * w \rangle$. Then it follows, for $u_1, u_2 \in U$ and $w \in W$, that $\langle u_1 * u_2, w \rangle = \langle u_1, u_2 * w \rangle$. But, $W$ is a two-sided ideal under convolution, so the latter expression is zero for all $w \in W$, but this can't be true for $w = u_1 * u_2$.) □

## 2.1 Homogeneous spaces and and spherical harmonics

Recall that one of our original sources of representations was from spaces $X$ with *nonlinear* action of a group $G$ by automorphisms—by linearising, such as taking $T_xX$ for $x \in X$ a fixed point, or taking cohomology $H^*(X)$.

A homogeneous space $X$ is one for which a group $G$ acts transitively by automorphisms. For example, the sphere $S^2$ has a transitive action by the group $\mathsf{SO}_3(\mathbb{R})$, and more generally $S^n$ has such an action of $\mathsf{SO}_{n+1}(\mathbb{R})$. For a homogeneous space $X$ and point $x \in X$, we get an isomorphism $G/H \to X, gH \mapsto g \cdot x$, for $H = G_x$ the stabiliser of $x$.

**Definition 2.4.** Given a representation $(V, \rho_V)$ of $G$ and a subgroup $H \leq G$, we define the $H$-invariant subspace to be $V^H := \{v \in V \mid \rho_V(h)(v) = v, \forall h \in H\}$.

Next observe that $L^2(G/H) \cong L^2(G)^H$. We can plug this into the Peter–Weyl theorem to deduce:

**Corollary 2.5.** *The representation $L^2(G/H)$ of $G$ is the closure of the orthogonal decomposition:*

$$\bigoplus_V V \otimes (V^*)^H,$$

*with $(V^*)^H \cong V^H$ via the inner product. In the case of $G = \mathsf{SO}_{n+1}(\mathbb{R}) \geq \mathsf{SO}_n(\mathbb{R}) = H$, we deduce that $L^2(S^n)$ is the closure of the orthogonal decomposition:*

$$\bigoplus_V V \otimes (V^*)^{\mathsf{SO}_n(\mathbb{R})}.$$

*In the particular case of $n = 2$, we get that $L^2(S^2)$ is the closure of the orthogonal decomposition*

$$\bigoplus_{m=0}^{\infty} V_{2m},$$

*with $V_{2m} \cong \mathbb{C}[x, y]_{2m}$ the $2m+1$ dimensional irreducible representation of $\mathsf{SO}_3(\mathbb{R}) = S^3/\{\pm 1\}$.*

This decomposition of $L^2(S^2)$ into irreducible representations of $\mathsf{SO}_3(\mathbb{R})$ is nothing but the decomposition into *spherical harmonics*.

# 3 The Borel–Weil–Bott theorem

Finally, we turn from $L^2$ functions on real groups to *polynomial* functions on complex algebraic groups. The general goal is to produce the representations from this setting, and also to gain insight into higher cohomology (a phenomenon which does not occur in the smooth setting).

So take the complex Lie group $\mathsf{SL}_2(\mathbb{C})$. Note that the representations we constructed $\mathbb{C}[x, y]_m$ can be viewed as functions on any homogeneous space $\mathsf{SL}_2(\mathbb{C})/H$ with $H$ acting trivially on $x$. In analogy with the spherical harmonics, it would be a good idea to take the maximal subgroup $B = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. The quotient $\mathsf{SL}_2(\mathbb{C})/B$ is nothing by $\mathbb{P}^1$. This is projective! (More generally, given a semisimple complex Lie group $G$ and maximal solvable subgroup $B$, the quotient $G/B$ has the structure of a projective variety.) However, this can't work as stated, since every algebraic function on a projective variety is trivial. This is even true for holomorphic functions, which by Serre's GAGA are anyway the same as algebraic functions when the space is projective. On the other hand, we can notice that $\mathbb{C}[x,y]_m = \Gamma(\mathbb{P}^1, \mathcal{O}(m))$. This suggests the following result:

**Theorem 3.1** (Borel–Weil)**.** *Let $G$ be a simply-connected complex algebraic semisimple Lie group and $B \leq G$ a maximal connected solvable subgroup. Then the irreducible (finite-dimensional) representations of $G$ are given as $\Gamma(G/B, \mathcal{L})$ for $\mathcal{L}$ ranging over all line bundles with nonzero spaces of global sections.*

**Remark 3.2.** Here there is a technical point arising because we use nontrivial line bundles: if $X$ admits an action of $G$, in order for $\Gamma(X, L)$ to also admit an action of $G$, we need the total space of $L$ to admit a compatible, fibrewise linear $G$-action (called a $G$-equivariant structure). In this case this structure exists and is unique. See the exercise sheet for more details on this. But if this did not hold, then

all we would have to do is modify the statement to say "$G$-equivariant line bundle" instead of just "line bundle". Actually, this is a better way to state the theorem, since it makes it clearer that the bundles we are considering are constructed from characters of $B$ (by the associated line bundle to the $B$-bundle $G \to G/B$, see below for an equivalent definition).

Thanks to the remark, all line bundles on $G/B$ are $G$-equivariant, therefore have the form $\overline{G \times \mathbb{C}_\chi B}$ for $\chi : B \to \mathbb{C}^\times$ a character of $B$. The data of $\chi$ is equivalent to a character of the abelianisation $B/[B,B]$, since $\mathbb{C}^\times$ is abelian, so $\chi([B,B]) = \{1\}$. By the structure theory of semisimple Lie groups, the latter is isomorphic to a maximal torus $T \leq G$, with $T \cong (\mathbb{C}^\times)^{r(G)}$ for $r(G) \geq 0$ called the *semisimple rank* of $G$. So we get $\mathsf{Pic}(G/B) \cong \mathsf{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^{r(G)}$.

**Example 3.3.** For the group $\mathsf{SL}_2(\mathbb{C})$, $B$ can be taken to be the upper-triangular matrices, $\left\{\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^\times, \mu \in \mathbb{C}\right\}$. So $\mathsf{Pic}(\mathbb{P}^1) = \{\mathcal{O}(m)\} \cong \mathbb{Z}$. The quotient $G/B \cong \mathbb{P}^1$, more concretely $G$ acts on $\mathbb{P}^1 = \{[a:b]\}$ with stabiliser of $[1:0]$ equal to $B$. We see that $B/[B,B] \cong \mathbb{C}^\times \cong \left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$. The semisimple rank is one, which indeed equals the Picard rank of $G/B \cong \mathbb{P}^1$.

As already observed, the global sections $\Gamma(\mathbb{P}^1, \mathcal{O}(m)) \cong \mathbb{C}[x,y]_m$ for $m \geq 0$, and this space is zero for $m < 0$. This proves the theorem in the case of $\mathsf{SL}_2(\mathbb{C})$, thanks to Example 1.26.

## 3.1 Extension to higher cohomology

Unlike the smooth case, taking global sections is not an exact functor working holomorphically (or algebraically). Explicitly, if $V \to L$ is a surjection from a vector bundle (or sheaf) $V$ to $L$, then $\Gamma(X, V) \to \Gamma(X, L)$ need not be surjective. In such a situation, one should consider not merely the global sections themselves, but also their derived functors $R^i\Gamma(X, L) =: H^i(X, L)$, the sheaf cohomology. In particular, for $X = \mathbb{P}^1$, note that

$\Gamma(\mathbb{P}^1, \mathcal{O}(m)) = 0$ for $m < 0$. On ther other hand, $H^1(\mathbb{P}^1(\mathcal{O})(m)) \neq 0$ for $m < -1$. It turns out that there is only one line bundle, $\mathcal{O}(-1)$, on $\mathbb{P}^1$, so that all the cohomology spaces are zero. A way to see this is via Serre duality: the canonical bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(-2)$, so we have $H^1(\mathbb{P}^1, \mathcal{O}(m)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-m-2))$.

This generalises to the following statement:

**Theorem 3.4** (Borel–Weil–Bott). *For every line bundle $L_\chi$ on $X = G/B$, there is at most one $i \in \mathbb{N}$ such that $H^i(X, L_\chi) \neq 0$, and this is an irreducible (finite-dimensional) representation of $G$.*

We can make this more precise, by specifying what the degree $i$ is and what the irreducible representation is, as well as when this $i$ exists. The answer expresses itself in terms of the combinatorics (root system) arising from the geometry and structure of $G$:

For $T \leq G$ a maximal torus, recall that the Weyl group $W_G$ is defined as $N_G(T)/T$, the normaliser of the torus modulo the torus. For example, if $G = \mathsf{SL}_n(\mathbb{C})$, then we can take $T$ to be the diagonal matrices of determinant one, and $W$ is the group of permutation matrices (or more precisely, the permutation matrices times $T$ modulo $T$). Then $W_G$ acts on $\mathsf{Pic}(G) \cong \mathbb{Z}^{r(G)}$. What is interesting is that we need a shifted version of this action: there is an element called $-\rho \in \mathbb{Z}^{r(G)}$ which generalises the bundle $\mathcal{O}(-1)$ in the case of $G = \mathsf{SL}_2(\mathbb{C})$. ($-\rho$ is one-half the sum of the positive roots, or equivalently the sum of all the fundamental weights; see the exercise sheet.) We then define the shifted action of $W$ on $\mathbb{Z}^{r(G)}$ by:

$$w \cdot v := w(v + \rho) - \rho, \tag{2}$$

i.e., the linear action with origin $-\rho$ instead of 0. Next, it is convenient to view the action of $W$ on $\mathbb{Z}^{r(G)}$ as on the real vector space $\mathbb{R}^{r(G)}$, which is a real representation of the finite group $W$. We have the following structural result on this representation:

**Theorem 3.5.** *The action of $W$ on $\mathbb{Z}^{r(G)}$ is generated by reflections about a set of linear hyperplanes $\mathcal{H} \subseteq \mathbb{R}^{r(G)}$, and $\mathbb{R}^{r(G)} \setminus \bigcup_{H \in \mathcal{H}} H$ is a union of $|W|$ connected cones, which are permuted simply transitively by the action of $W$.*

**Definition 3.6.** Fixing one of these cones $C$ which we call the *dominant* cone, to every other cone $w(C)$ assign the number $N_{w(C)} \geq 1$ which is the number of hyperplanes in $\mathcal{H}$ separating $w(C)$ from $C$. We call this the *length* of $w$, denoted also by $\ell(w)$.

**Remark 3.7.** The action of the maximal torus $T$ on the the Lie algebra $\mathfrak{g}$ of $G$ under the adjoint action decomposes $\mathfrak{g}$ into weight spaces, given by a subset of $\mathsf{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^{r(G)}$, the nonzero elements of which are called the *roots*, and denoted by $\Phi \subseteq \mathbb{Z}^{r(G)}$. Each root $\alpha$ is naturally associated with a "coroot" $\alpha^\vee$ of the dual space, whose complexification is the complex Lie algebra $\mathfrak{h}$ of $T$, also called a Cartan subalgebra; and the hyperplanes are the annihilators $(\alpha^\vee)^\perp$. This also defines a natural inner product on the space of roots, given by $\langle \alpha, \beta \rangle = \alpha^\vee(\beta)$.

**Example 3.8.** For $G = \mathsf{SL}_n(\mathbb{C})$, we have $r(G) = n - 1$, the group $W$ is $S_n$, and the vector space $\mathbb{R}^{r(G)}$ above is the (real) reflection representation $V_{\text{refl}}$ of $S_n$, defined as the subspace

$V_{\text{refl}} \subseteq \mathbb{R}^n$ of vectors summing to zero, $\mathbb{R}^{n-1} \cong \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1 + \cdots + a_n = 0\}$. The set of hyperplanes $\mathcal{H}$ then identifies with the locus where two coordinates are equal, $H_{ij} = \mathbb{R}^{n-1} \cap \{(a_1 \ldots, a_n) \mid a_i = a_j\}$, with reflection the transposition $(ij)$. The length of a permutation $\sigma \in S_n$ is given by $\ell(\sigma) = |\{(i,j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}|$. Note that you have all seen this in the definition of the sign of a permutation: $\mathsf{sign}(\sigma) = (-1)^{\ell(\sigma)}$.

We can now finally state the precise version of the Borel–Weil–Bott theorem:

**Theorem 3.9.** *The cohomology $H^\bullet(X, L_\chi)$ is zero if and only if $\chi + \rho$ lies on a hyperplane of $W$ on $\mathbb{Z}^{r(G)}$. The global sections $\Gamma(X, L_\chi)$ are nonzero if and only if $\chi \in C$ for a particular chamber $C$ of the hyperplane complement $\mathbb{R}^{r(G)} \setminus \bigcup_{H \in \mathcal{H}} H$. Letting this one be the dominant chamber, for every $w \in W$ and $\chi \in w(C)$, we have*

$$H^i(X, L_\chi) = \begin{cases} \Gamma(X, L_{w^{-1}(\chi)}), & i = \ell(w), \\ 0, & i \neq \ell(w). \end{cases}$$

A way this is often stated is that, if $w \in W$ is the reflection about a hyperplane $H$ which is the boundary between two components $C', C''$ of the complement of $\bigcup \mathcal{H} \subseteq \mathbb{R}^{r(G)}$, then for $\chi \in C'$, we have $H^i(X, L_\chi) \cong H^{i \pm 1}(X, L_{w(\chi)})$, with the sign positive if $C'$ is on the same side of $H$ as the dominant chamber $C$.

## 3.2 Example: $\mathsf{SL}_3(\mathbb{C})$

In the case of $G = \mathsf{SL}_3(\mathbb{C})$, the maximal torus is taken to be $T = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & (ab)^{-1} \end{pmatrix} \right\}$, of dimension $r(G) = 2$. The space $\mathbb{R}^2 \subseteq \mathbb{R}^3$ is the locus of triples $\{(a,b,c) \mid a+b+c = 0\}$, and the hyperplanes there are the three lines $\{(a,a,b)\}, \{(a,b,a)\}$, and $\{(b,a,a)\}$. The shifted action is centred at $-\rho = (-1, 0, 1)$. The usual way to draw this in two-dimensional space is as the horizontal axes and the axes making $60°$ angles with the horizontal, with the lattice spanned by the vectors $(1,-1,0)$ and $(0,1,-1)$, viewed as the length one vectors in the direction $-30°$ perpendicular to the $60°$ hyperplane, and the direction $90°$ ($y$-axis direction), perpendicular to the horizonal hyperplane. I drew the rough picture on the board at the end of the lecture.